

GTSG: Characteristic Classes

Riley Moriss

April 30, 2026

1	Definitions	1
1.1	Fixing Functors	2
1.2	On Numbers	2
2	Existence Proofs	3
2.1	Stiefel-Whitney Class	3
2.2	Chern Class	3
2.3	Pontryagin Class	4
3	Propositions more or less from the axioms	4
4	Calculuations	5
4.1	Sphere	5
4.2	Complex Projective Space	5

1 Definitions

Lets start with something abstract and then unpack it, we will be intentionally vague. **A characteristic class** in general can be defined as a natural transformation between two functors, one is a cohomology theory and one is a relevant theory of bundles (map on morphisms is pulling back),

$$H^* : \text{Spaces}^{\text{op}} \rightarrow \mathcal{C}$$

$$\text{Bund}_- : \text{Spaces}^{\text{op}} \rightarrow \mathcal{C}$$

For a natural transformation to make sense the domain and codomain of the functors has to match. It is understood that cohomology can produce different things, and so candidates for \mathcal{C} are therefore Set, Rings, AbelianGroups or Monoids (associative and identity). We would need to endow our vector bundle functor with the same structure however, as a priori it just gives the *set* of bundles.

A natural transformation is then a family of morphisms for any space B

$$\text{Bund}(B) \rightarrow H^*(B)$$

that satisfy the naturality conditions. In particular if we have a morphism of spaces $f : N \rightarrow M$, we need the following diagram to commute

$$\begin{array}{ccc} \text{Bund}(M) & \xrightarrow{\text{Vect}(f)} & \text{Bund}(N) \\ \downarrow c(M) & & \downarrow c(N) \\ H^*(M) & \xrightarrow{H^*(f)} & H^*(N) \end{array}$$

the commutativity of this diagram explicitly says that

$$f^*c(\xi) = c(f^*\xi).$$

Note that the morphisms $c(M)$ are in \mathcal{C} and so if this is say monoids, then they must be monoid homomorphisms.

Remark: The difference between H^Π and H^* as rings is that H^Π has no finiteness condition, that is H^* is the *direct sum* of the cohomology groups and hence any element in the direct sum will have only finitely many non-zero components. This is not the case for H^Π . There is no difference for finite CW complexes.

1.1 Fixing Functors

From now on we will be considering the following functors

$$H^* : \text{SmoothManifolds}^{\text{op}} \rightarrow \text{Monoids}$$

$$\text{Vect}_- : \text{SmoothManifolds}^{\text{op}} \rightarrow \text{Monoids}$$

Cohomology is given the monoid structure under *cup product* and vector bundles is given that of *Whitney sum* (pulling back over the same base). The fact that our natural transformations are monoid homomorphisms then translates to

$$c(M)(\xi \oplus \tau) = c(M)(\xi) \smile c(M)(\tau)$$

or more commonly we suppress the natural transformation notation when the base is clear

$$c(\xi \oplus \tau) = c(\xi) \smile c(\tau).$$

Note that cohomology factors through graded monoids and we will use this fact, however we do not want the natural transformations to be between graded functors, because we want say rank n bundles to have potentially non-trivial rank $n - 1$ characteristic classes etc. We write $c_i(M)$ for the composition of $c(M)$ with the projection of $H^* \rightarrow H^i$.

It is common to enforce two normalising conditions on the characteristic classes of this form to ensure uniqueness. We assume that

$$c_0(\xi) = 1 \in H_0(M), \text{ for any bundle } \xi$$

and second we assume that c_1 of some canonical bundle of some canonical space is a generator. These are the axioms in [MS16] and [Hir66].

1.2 On Numbers

If we have a closed and oriented manifold M^n then there is a ‘canonical’ element in $H_n(M)$, namely the fundamental class, $[M]$. This gives us a some what natural way to assign to a manifold M a function $H^n(M; G) \rightarrow G$ given by evaluating a cohomology class on the fundamental class. Now there is also a canonical element of $\text{Vect}_-(M^n)$, namely the tangent bundle, and so we can produce elements of top degree cohomology by evaluating our characteristic classes on the tangent bundle. More than that we can get them by cupping up lower degree characteristic classes. Thus given a partition i_1, \dots, i_r of n , the dimension of the manifold and therefore the rank of the tangent bundle, we can get

$$c_{i_1}(\tau_M) \cdots c_{i_r}(\tau_M)[M] \in G$$

The partition just ensures that the product is in the correct degree of cohomology, any such product that lands in the right degree is valid. These are called the characteristic numbers and clearly depend on the given partition. It is a fact that in many cases the characteristic numbers are sufficient information and the classes are not needed.

2 Existence Proofs

We would like to actually construct a characteristic class, and perhaps thereby intuit their geometric meaning.

2.1 Stiefel-Whitney Class

These are natural transformation $\text{Vect}_{\mathbb{R}}(-) \Rightarrow H^*(-; \mathbb{Z}/2\mathbb{Z})$, from the monoid of real vector bundles to the monoid of mod 2 cohomology. We further normalise assuming that c_1 of the mobius bundle over the circle is a generator.

Lemma. *Stiefel-Whitney classes exist.*

They are defined abstractly as the composition of some important operations that we dont fully understand, let $\xi : E \rightarrow B$ be a rank n bundle:

$$w_i(\xi) := \varphi^{-1} \text{Sq}^i \varphi(1)$$

where $\varphi : H^k(B) \rightarrow H^{k+n}(E, E_0)$ is the Thom isomorphism and $\text{Sq}^i : H^k(E, E_0) \rightarrow H^{k+i}(E, E_0)$ is the ‘Steenrod squaring’ operation. We set $k = 0$ to get a map $H^0(B) \rightarrow H^i(B)$ and since every manifold is $\mathbb{Z}/2\mathbb{Z}$ orientable we have a well defined element $1 \in H^0(B; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ (using Poincare duality with the top degree homology **Can I see this without Poincare duality? — We will have to offer another talk about power operations..**).

An alternate perspective would be to recognize that any rank n vector bundle $E \rightarrow B$ is classified by a map $f : B \rightarrow BO_n$ by pulling back the universal bundle $EO_n \rightarrow BO_n$. Milnor-Stasheff give a model for BO_n in what they call the infinite Grassman manifold G_n whose points are n -planes in \mathbb{R}^∞ , the universal bundle they model as a tautological bundle over this space. Then given a cohomology class $c \in H^*(BO_n; \mathbb{Z}/2\mathbb{Z})$ we can pull it back along the classifying map to obtain a class $f^*c \in H^*(B; \mathbb{Z}/2\mathbb{Z})$. Thus to define a characteristic class it is enough to specify an element of $H^*(BO_n; \mathbb{Z}/2\mathbb{Z})$. Milnor-Stasheff compute this cohomology in §7 where they give that

$$H^*(BO_n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{F}_2[x_1, \dots, x_n]$$

as rings, that is the free polynomial algebra on n generators. The proof proceeds by first giving an explicit cell structure on the Grassman manifolds, showing that the generators dont satisfy any relations and that for dimensional reasons they span. The universal Stiefel-Whitney classes are then the generators, but this would also work as a definition. [McC00, Thm 6.38] computes this cohomology using the Serre spectral sequence.

2.2 Chern Class

This is a natural transformation $\text{Vect}_{\mathbb{C}}(-) \Rightarrow H^*(-; \mathbb{Z})$, from the monoid of complex vector bundles to the monoid of integral cohomology. We normalise by assuming that c_1 of the canonical bundle over $\mathbb{C}P^n$ (fiber at a point is the point of $\mathbb{C}P^n$ which is a line) is non-trivial, that is it is a generator of $H^*(\mathbb{C}P^n)$.

Lemma. *Chern classes exist.*

Complex vector bundles are classified by maps into BU_n . [MS16, Thm 14.5] The integral cohomology of BU_n is known to be

$$H^*(BU_n; \mathbb{Z}) \cong \mathbb{Z}[x_1, \dots, x_n]$$

the polynomial ring on n generators. Thus we can define the Chern classes as the pullbacks of these generators.

2.3 Pontryagin Class

This is a natural transformation $\text{Vect}_{\mathbb{R}}(-) \Rightarrow H^*(-; \mathbb{Z})$, from the *set* of complex vector bundles to the *set* of integral cohomology. We normalise by the formula relating these classes to the Chern classes. As it turns out this means assuming that c_1 of the canonical bundle over $\mathbb{C}P^n$ is the square of the generator of H^2 . Notice that we have made this only a natural transformation between functors that land in Sets. This is because Pontryagin classes are not additive on the nose.

Lemma. *Pontryagin classes exist.*

These are the easiest to state because all the work was done in the case of Chern classes, we define

$$p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C})$$

Of course we should now check that they satisfy all the properties we want, but they mostly follow immediately from the Chern class results. Moreover they do have some additivity properties, namely mod two

Lemma. $2(p(\xi \oplus \xi') - p(\xi) \smile p(\xi')) = 0$.

Returning to the universal bundle perspective [Bro82] it has been known for a long time that

$$H^*(BO_n; \mathbb{Z}) \cong \mathbb{Z}[p_1, \dots, p_{n/2}] \oplus T$$

where T is **applying the Bockstein to the Stiefel-Whitney classes or something**. Brown gives generators and relations. Thus we could have in principle defined the Pontryagin classes as the pullbacks of the integral generators in the cohomology of BO_n .

3 Propositions more or less from the axioms

Here are some key properties

1) The characteristic class of a trivial bundle is 1.

Consider a trivial bundle $B \times \mathbb{R}^n \rightarrow B$ then consider the map $\bar{f} : B \rightarrow *$. This induces a map $f : B \times \mathbb{R}^n \rightarrow * \times \mathbb{R}^n$ which is clearly a bundle map, to the trivial bundle over a point. Now apply naturality of the characteristic classes we get that

$$c_i(B \times \mathbb{R}^n) = \bar{f}^* c_i(* \times \mathbb{R}^n) \in H^i(*)$$

and so must be zero for $i > 0$ and 1 in degree 0.

2) Isomorphic bundles (over the same base) have equal characteristic classes (more precisely their characteristic classes map to one another under the given isomorphism).

This is non-trivial as far as I can see. One first needs to prove that these classes exist and are unique. Given this however it might follow from the fact that isomorphisms induce isomorphisms in cohomology.

3) Stiefel-Whitney classes commute with (exterior) products:

$$w(\xi \times \xi') = w(\xi) \times w(\xi')$$

where the product on the right is the exterior product and on the left is the ‘‘Cartesian product’’, which is a complex construction from pulling back pullbacks. Naively this can just be considered as a formal product of the cohomology classes, where the coefficients commute but the generators of the cohomology rings are different formal variables.

4) Chern classes and conjugation are well behaved:

$$c_k(\bar{\xi}) = (-1)^k c_k(\xi)$$

4 Calculations

4.1 Sphere

The first easiest example is that of the sphere. We know that for the standard embedding

$$S^n \hookrightarrow R^{n+1}$$

the normal bundle is trivial, ϵ , thus applying the product rule we have that

$$1 = w(\epsilon) = w(\tau_{S^n} \oplus \nu) = w(\tau_{S^n})w(\nu) = w(\tau_{S^n}).$$

4.2 Complex Projective Space

This is a complex manifold with a complex tangent bundle. If we compute the Chern classes then we will know also the Euler class and the Steiffel-Whitney classes.

If we consider the model of $\mathbb{C}P^n$ as

$$\mathbb{C}P^n := \{L \leq \mathbb{C}^n : \text{rank 1 subspaces}\}$$

If we define the bundles

$$\begin{aligned} \gamma &:= \{(v, L) : v \in L\} \subseteq \mathbb{C}^{n+1} \times \mathbb{C}P^n \\ \gamma^\perp &:= \{(v, L) : v \in L^\perp\} \subseteq \mathbb{C}^{n+1} \times \mathbb{C}P^n \end{aligned}$$

with the projection from the second variable. Note that γ is called *the canonical bundle*. The tangent space of $\mathbb{C}P^n$ has the elegant expression in the form

$$\tau = \tau_{\mathbb{C}P^n} = \text{Hom}(\tau, \tau^\perp)$$

where we consider Hom the bundle constructed fiberwise. Intuitively an element of this hom bundle is a map from the fiber at L which is just L to the normal to L , but L is one dimensional and so this is just an element of the normal space, i.e. a tangent vector. **Actually would like to understand how is the space of lines topologised and how is that related to the cell structure.**

We also know that $\text{Hom}(\gamma, \gamma) \cong \epsilon$ that is a trivial bundle, because it has a no-where zero section given by the identity map at each point (thus it is parrallelizable) and is of dimension 1, each fiber is the point over which it is fiber, which are by definition rank 1 subspaces. Using that the trivial bundle has no effect on Chern number we get that

$$\begin{aligned} c(\tau) &= c(\tau \oplus \epsilon) \\ &= c(\tau \oplus \text{Hom}(\gamma, \gamma)) \\ &= c(\text{Hom}(\gamma, \gamma^\perp) \oplus \text{Hom}(\gamma, \gamma)) \\ &= c(\text{Hom}(\gamma, \gamma \oplus \gamma^\perp)) \end{aligned}$$

But $\gamma \oplus \gamma^\perp$ is a trivial rank $n + 1$ bundle and so we get that

$$\begin{aligned} c(\tau) &= c(\text{Hom}(\gamma, \mathbb{C}^{\oplus n+1})) \\ &= c(\text{Hom}(\gamma, \mathbb{C})^{\oplus n+1}) \\ &= c(\bar{\gamma}^{\oplus n+1}) \\ &= c(\bar{\gamma})^{n+1} \end{aligned}$$

The $n + 1$ fold cup product. Thus we get a binomial expansion in terms of the Chern class of this γ bundle. By our 4th axiom this is a generator of $H^2(\mathbb{C}P^n)$.

Now lets compute some example numbers of these spaces (the general case is in [MS16] but is just combinatorics from these basic ones), denote $\alpha = c_1(\bar{\gamma})$, then for example we know that

$$c(\mathbb{C}P^2) = (1 + \alpha)^3 = 1 + 3\alpha + 3\alpha^2$$

noting that the final term α^3 from the binomial expansion is in $H^6(\mathbb{C}P^2) = 0$. α^2 is dual to the fundamental class of $\mathbb{C}P^n$, by convention it is dual on the nose, without a $-$ sign (they are both generators of the respective rank one modules and are therefore dual). Thus we can compute the Chern numbers by considering partitions of 4 by even numbers (Chern classes only in even degrees), of which there are two $2 + 2$ and 4. These correspond to the Chern classes

$$c_1^2 = (3\alpha)^2, \quad c_2 = 3\alpha^2$$

which we can evaluate on $[\mathbb{C}P^2]$ and use its duality with α^2 to get immediately the associated Chern numbers

$$9, 3$$

respectively.

Now $\mathbb{C}P^1 \times \mathbb{C}P^1$ has Chern class

$$c(\mathbb{C}P^1 \times \mathbb{C}P^1) = c(\mathbb{C}P^1) \times c(\mathbb{C}P^1) = (1 + \alpha)^2(1 + \beta)^2 = (1 + 2\alpha)(1 + 2\beta) = 1 + 2\alpha + 2\beta + 4\alpha\beta$$

it is of the same dimension, 4 so we have the same partitions and we can again compute two Chern numbers. By looking at the degree of cohomology the terms are in we recover which Chern charater the parts of the sum correspond to and we get

$$c_1^2 = (2\alpha + 2\beta)^2 = 4\alpha^2 + 8\alpha\beta + 4\beta^2 = 8\alpha\beta, \quad c_2 = 4\alpha\beta$$

the squares of the cohomology classes are zero for dimensional reasons. Now we again have by convention a duality between $\alpha\beta$ and the fundamental class thus the two respective numbers are

$$8, 4.$$

Lets go up a dimension and compute Pontryagin classes and numbers. If we didnt go up a couple of dimensions (4) there would only be one number, not very interesting. Lets do similar examples of the 8 manifolds $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$. A key fact is that for a complex manifold with a complex tangent space we have that

$$\tau \otimes \mathbb{C} \cong \tau \oplus \bar{\tau}$$

thus using the product property and denoting the tangent bundle of $\mathbb{C}P^4$ as τ we get that

$$\tilde{p}(\tau) = c(\tau)c(\bar{\tau}) = (1 + \alpha)^5(1 - \alpha)^5 = (1 - \alpha^2)^5 = 1 - 5\alpha^2 + 10\alpha^4 - 10\alpha^6 + 5\alpha^8 - \alpha^{10} = 1 - 5\alpha^2 + 10\alpha^4$$

where the higher terms are zero for dimension reasons. In otherwords

$$p(\tau) = 1 + 5\alpha^2 + 10\alpha^4$$

We need partitions of 8 that are in degrees 4 and 8 of which there are again (essentially the same) thus we have the two classes of interest

$$p_1^2 = 25\alpha^4, \quad p_2 = 10\alpha^4$$

which gives the numbers 25, 10.

For $\mathbb{C}P^2 \times \mathbb{C}P^2$ we have the Pontryagin class of the tangent space given by

$$\tilde{p}(\tau) = \tilde{p}(\tau_{\mathbb{C}P^2}) \times \tilde{p}(\tau_{\mathbb{C}P^2}) = (1 + \alpha)^3(1 - \alpha)^3(1 + \beta)^3(1 - \beta)^3 = (1 - 3\alpha^2)(1 - 3\beta^2) = 1 - 3(\alpha^2 + \beta^2) + 9\alpha^2\beta^2$$

Again in other words we have that

$$p(\tau) = 1 + 3(\alpha^2 + \beta^2) + 9\alpha^2\beta^2$$

The two classes are then

$$p_1^2 = 9(\alpha^2 + \beta^2)^2 = 9(\alpha^4 + 2\alpha^2\beta^2 + \beta^4) = 18\alpha^2\beta^2, \quad p_2 = 9\alpha^2\beta^2$$

thus the numbers are 18, 9 respectively.

	$\mathbb{C}P^2$	$\mathbb{C}P^1 \times \mathbb{C}P^1$
c_1^2	9	8
c_2	3	4

	$\mathbb{C}P^4$	$\mathbb{C}P^2 \times \mathbb{C}P^2$
p_1^2	25	9
p_2	10	1

References

- [Bro82] Edgar H. Brown. The Cohomology of BSO_n and BO_n with Integer Coefficients. *Proceedings of the American Mathematical Society*, 85(2):283, June 1982.
- [Hir66] Friedrich Hirzebruch. *Topological Methods in Algebraic Geometry*. Springer, Berlin, Heidelberg, 1966.
- [McC00] John McCleary. *A User's Guide to Spectral Sequences*. Cambridge University Press, 2 edition, November 2000.
- [MS16] John Willard Milnor and James D. Stasheff. *Characteristic Classes: AM-76*. Number 76 in Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2016.